International Journal of Theoretical Physics, Vol. 40, No. 10, October 2001 (© 2001)

Generation and Properties of a Superposition of Four Displaced Fock States

A.-S. F. Obada[1,2] and G. M. Abd Al-Kader[1]

Received October 17, 2000; accepted May 14, 2001

A detailed discussion of a type of four-component superposition of displaced Fock states (DFSs) is presented. A generation scheme is proposed for these states. The *s*-parameterized characteristic function (CF) and the quasiprobability distribution functions (QDFs) of these states are calculated. The nonclassical properties of these states such as photon number distribution and squeezing are discussed. The quadrature distributions are illustrated. The Pegg–Barnett phase distribution is discussed.

1. INTRODUCTION

The concept of the photon in the quantum theory of a radiation field has been built on the Fock (number) state $|n\rangle$. However, the coherent state is another important state; it may be defined by the action of a displacement operator $D(\alpha)$ on the vacuum state. These states have been extensively studied (Glauber, 1963; Perina, 1984; Walls and Milburn, 1994). On the other hand, the displaced Fock states (DFSs) are very important kinds of states in quantum optics, defined by the action of the displacement operator on the number state (Abd Al-Kader, 1994; Agarwal and Tara, 1991; Boiteux and Levelut, 1973; De Oliveira *et al.*, 1990; Moya-Cessa and Knight, 1993; Roy and Singh, 1982; Satyanarayana, 1985; Wunsche, 1991). They can be regarded as a generalized class of the Fock and coherent states. They form a complete basis, and have interesting and unusual physical properties (Abd Al-Kader, 1994; Agarwal and Tara, 1991; Boiteux and Levelut, 1973; De Oliveira *et al.*, 1990; Moya-Cessa and Knight, 1993; Roy and Singh, 1982; Satyanarayana, 1985; Wunsche, 1991). The quasiprobability distribution functions (QDFs) have been represented as a series in terms of these states (Abd Al-Kader, 1994; Agarwal and Tara, 1991; De Oliveira *et al.*, 1990; Moya-Cessa

[1] Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt.
[2] To whom correspondence should be addressed at Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt; e-mail: obada@mailer.scu.eun.eg.

and Knight, 1993; Wunsche, 1991). Experiments have been performed to prepare the Fock states, coherent states, and states derived from them, in recent years (de Matos Filho and Vogel, 1996; Gardiner *et al.*, 1997; Itano *et al.*, 1997; Kneer and Law, 1998; Law and Eberly, 1996; Monroe *et al.*, 1996; Steinbach *et al.*, 1997; Vogel and de Matos Filho, 1995; Wineland *et al.*, 1998). The various schemes proposed have been built on the motional dynamics of the centre of mass of trapped ions (Abd Al-Kader, 1999; Obada and Abd Al-Kader, 1998, 1999; Schleich and Raymer, 1997).

Recently, a considerable effort has been devoted to a description of the nonclassical properties of superpositions of quantum states of light (Abd Al-Kader, 1999; Abdalla *et al.*, 1994; Ban, 1995; Bose *et al.*, 1997; Buzek *et al.*, 1992; Dakna *et al.*, 1997, 1998; Dodonov *et al.*, 1996, 1998; Garraway *et al.*, 1994, 1995; Gou *et al.*, 1997; Hach and Gerry, 1992, 1993; Lee *et al.*, 1993; Mogilevtsev and Ya, 1996; Moya-Cessa, 1995; Obada and Abd Al-Kader, 1998, 1999; Obada and Omar, 1995, 1997; Schleich and Raymer, 1997; Yurke and Stoler, 1986; Zheng and Guo, 1996). Squeezing is one of many nonclassical properties that originate from the quantum interference between the component states of the superposition states. In addition, there has been much interest in the properties and in the generation of the various superposition states of light (Abd Al-Kader, 1999; Abdalla *et al.*, 1994; Ban, 1995; Bose *et al.*, 1997; Buzek *et al.*, 1992; Dakna *et al.*, 1997, 1998; Dodonov *et al.*, 1996, 1998; Garraway *et al.*, 1994, 1995; Gou *et al.*, 1997; Hach and Gerry, 1992, 1993; Lee *et al.*, 1993; Mogilevtsev and Ya, 1996; Moya-Cessa, 1995; Obada and Abd Al-Kader, 1998, 1999; Obada and Omar, 1995, 1997; Schleich and Raymer, 1997; Yurke and Stoler, 1986; Zheng and Guo, 1996). Schrödinger cat states are quantum superpositions of macroscopically distinguishable states, they can be produced in quantum optical experiments (Abd Al-Kader, 1999; Obada and Abd Al-Kader, 1998, 1999; Schleich and Raymer, 1997). Since the first proposal made by Yurke and Stoler (1986), a lot of interest has been paid to the idea of using nonlinear wave-mixing processes for the generation of multicomponent entangled Schrödinger cat states of an electromagnetic field (Ban, 1995; Buzek *et al.*, 1992; Garraway *et al.*, 1994, 1995; Yurke and Stoler, 1986). Creation of the superposition states by applying a sequence of laser pulses, which entangle internal (electronic) and external (motional) states of the ion, have been reported in Schleich and Raymer (1997), Obada and Abd Al-Kader (1998, 1999), and Abd Al-Kader (1999). They showed that by choosing appropriate interaction durations, a coherent input state can be transformed into a superposition state of two or four coherent component states located on a circle (Bose *et al.*, 1997; Hach and Gerry, 1992; Lee *et al.*, 1993; Mogilevtsev and Ya, 1996). These fundamental researches not only lead us to a deeper understanding of the nature of light, but also have applications in the quantum communications and in detection of weak signals.

The characteristic function (CF) plays the central role in the fundamental exposition of the quasiprobability distribution functions (QDFs). It is defined as the trace of the product of the density operator with the displacement operator (Hillery *et al.*, 1984, and references therein; Lee, 1995). Different orders of product of creation and annihilation average values have been obtained by these functions. Also photon number distributions and various moments can be generated from this function (Cahill and Glauber, 1969; Hillery *et al.*, 1984; Lee, 1995; Wünsche, 1998).

Recently Barnett and Pegg (1986, 1989a,b) introduced a new Hermitian phase formalism that successfully overcomes the troubles inherent in the Susskind–Glogower (Carruthers and Nieto, 1968; Jackiw, 1968; Loudon, 1973; Susskind and Glogower, 1964) phase formalism and enables one to study finer details of the phase properties of quantum fields. Such quantities as expectation values and variances of the Hermitian phase operators or phase distribution functions are now available for investigations ("Quantum phase and phase dependent measurements," 1993; Lynch, 1995; Perinova *et al.*, 1998). One of our interests is to investigate the phase properties here.

In a previous paper we have considered the superposition of a pair of DFSs (Abd Al-Kader, 1999; Obada and Abd Al-Kader, 1998, 1999). These exhibit oscillations in the photon number distributions and other nonclassical properties, such as squeezing or sub-Poissonian photon statistics. A generation scheme for these states is presented by using the quantized motional degrees of freedom of a trapped ion. In this contribution we study the statistical properties of a multi-component DFS. To investigate the statistical properties of these multicomponent states, we evaluate the corresponding *s*-parameterized characteristic function (CF) and *s*-parameterized quasiprobability function (QDF).

In Section 2 a generation scheme for the so-called four-component superposition displaced Fock states (DFSs) is considered. In Section 3 we introduce the CF for the four component DFSs. In Section 4 the nonclassical properties are discussed, such as photon number distribution, correlation function, squeezing, phase distribution, quadrature distribution, and quasiprobability function. Finally we draw some conclusions.

2. GENERATION SCHEME OF A FOUR-COMPONENT SUPERPOSITION OF DFSs

Recent advances in the laser cooling and trapping of ions have made possible a realization of the Jaynes–Cummings model (Shore and Knight, 1993, and references therein), for which the usual single mode quantized electromagnetic field is replaced by the quantized vibrational motion of the ion's center of mass, which is coupled to two internal states of the ion by a classical driving laser field. In addition, it has become possible to generate a variety of motional states for the ion. These include thermal, Fock (number), and coherent (Schleich and Raymer, 1997), where the displaced Fock (number) and coherent states have distinctly nonclassical properties. The most general states of the pervious—the DFSs—have also been discussed in the context of trapped ions (Abd Al-Kader, 1999; Abdalla *et al.*, 1994; Ban, 1995; Bose *et al.*, 1997; Buzek *et al.*, 1992; Dakna *et al.*, 1997, 1998; Dodonov *et al.*, 1996, 1998; Garraway *et al.*, 1994, 1995; Gou *et al.*, 1997; Hach and Gerry, 1992, 1993; Lee *et al.*, 1993; Mogilevtsev and Ya, 1996; Moya-Cessa, 1995; Obada and Abd Al-Kader, 1998, 1999; Obada and Omar, 1995, 1997; Schleich and Raymer, 1997; Yurke and Stoler, 1986; Zheng and Guo, 1996).

The Jaynes-Cummings model (Shore and Knight, 1993, and references therein) is realized in the ion trap by the application of a laser tuned to the first upper vibrational sideband. Tuning to the first lower vibrational sideband yields the counterrotating terms in Jaynes-Cummings model. Tuning to the kth sidebands gives rise to k-photon vibrational analogue of multiphoton generalizations to the Jaynes–Cummings discussed in the quantum optics literature. However, driving only the resonant transition between the two levels of the ion results in a Kerr-type interaction. This is useful in making quantum nondemolition measurements of an ion and in generating quantum superposition of coherent states. In this section, we wish to consider the production of four-component states. Let a two-level ion of mass M move in a harmonic potential of frequency ω_x in the x-direction. Let $a(a^+)$ stand for the annihilation (creation) operator of the vibrational boson quanta in the x-direction. Then the position operator is given by $x = \Delta x_0(a + a^+)$, with $\Delta x_0 = (2\omega_x M)^{-1/2}$, the width of the harmonic ground state. In this scheme two beams of lasers applied along the x-axis are required to manipulate the motion of the atom: they are detuned by $\pm\omega_x$. In the rotating wave approximation the Hamiltonian for this system is given by

$$\mathbf{H} = \omega_x a^+ a + \frac{\omega_0}{2}\sigma_z - (\mu E^-(x, t)\sigma_- + h \cdot c). \tag{2.1}$$

The first two terms describe the external and internal free motion of the ion, and the last term stands for the atom–field interaction. The dipole matrix element μ and the transition frequency ω_0 of the two-level ion, and the operators $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma_+ = |e\rangle\langle g|$, $\sigma_- = |g\rangle\langle e|$, where $|e\rangle$ and $|g\rangle$ are the atomic excited and ground state respectively. The negative frequency part of the driving electric field is given by

$$E^-(x, t) = E_1\, e^{i[(\omega_0 - \omega_x)t - k_1 x + \phi_1]} + E_2\, e^{i[(\omega_0 + \omega_x)t - k_2 x + \phi_2]}, \tag{2.2}$$

where E_i and ϕ_i indicate amplitudes and phases of the driving beams. When the trapping frequency is much larger than the other characteristic frequencies, and providing that the field is resonant with one of the vibrational sidebands, then the ion–field interaction can be described by a nonlinear Jaynes–Cummings model (JCM) (de Matos Filho and Vogel, 1996; Gardiner *et al.*, 1997; Itano *et al.*, 1997; Kneer and Law, 1998; Law and Eberly, 1996; Monroe *et al.*, 1996; Schleich and Raymer, 1997; Steinbach *et al.*, 1997; Vogel and de Matos Filho, 1995; Wineland *et al.*, 1998). Accordingly, in the interaction picture the Hamiltonian (2.1) takes the form

$$\mathbf{H_I} = -\sum_{j=0}^{\infty}\left\{\Omega_1 e^{i\phi_1} e^{-\eta_1^2/2}\frac{(i\eta_1)^{2j+1}}{j!(j+1)!}(a^+)^{j+1}a^j \right.$$
$$\left. + \Omega_2 e^{i\phi_2} e^{-\eta_2^2/2}\frac{(i\eta_2)^{2j+1}}{j!(j+1)!}(a^+)^j a^{j+1}\right\}\sigma_- + h\cdot c. \tag{2.3}$$

$\Omega_j = \mu E_j$ are the Rabi frequencies and $\eta_l^2 = (k_l^2/2M)(1/\omega_x)$ are the Lamb–Dicke parameters, and they describe the ratio between the single photon recoil energy and the energy-level spacing in the harmonic oscillator strength. In the Lamb-Dicke limit where the vibrational amplitude of the ion is much smaller than the laser wavelength it is sufficient to keep the first few terms in (2.3), and one works with an effective Hamiltonian $\acute{\mathbf{H}}_\mathbf{I}$ of the form

$$\acute{\mathbf{H}}_\mathbf{I} = -(2g_1 a^+ + 2g_2 a)\sigma_- + h\cdot c. \tag{2.4a}$$

where

$$g_j = i\Omega_j e^{i\phi_j}\eta_j^2 e^{-\eta_j^2/2}, \quad j = 1, 2. \tag{2.4b}$$

The exponentials may be put equal to 1 because of the smallness of the η_j^2s.

In (2.4) the first term (+ its $\mathrm{h}\cdot\mathrm{c}$) is the usual JCM Hamiltonian. It describes the first red-side band resonance, while the second term (+ its $\mathrm{h}\cdot\mathrm{c}$) is the first blue-side band resonance. It is the counter-rotating term that is not present in the cavity Q.E.D. The motional and electronic dynamics may be decoupled in the Hamiltonian (2.4) by adding another interaction (de Matos Filho and Vogel, 1996; Gardiner *et al.*, 1997; Itano *et al.*, 1997; Kneer and Law, 1998; Law and Eberly, 1996; Monroe *et al.*, 1996; Steinbach *et al.*, 1997; Vogel and de Matos Filho, 1995; Wineland *et al.*, 1998), and we finish up with

$$\bar{\mathbf{H}}_\mathbf{I} = -\{(2(g_1 + g_2^*)a^+ + 2(g_1^* + g_2)a\}(\sigma_+ + \sigma_-). \tag{2.5}$$

Under this Hamiltonian any atom prepared in the state $(1/\sqrt{2})(|e\rangle + |g\rangle)$ that can be generated from the ground state by applying a $\pi/2$ carrier pulse will stay in this state and will be left unchanged (de Matos Filho and Vogel, 1996; Gardiner *et al.*, 1997; Itano *et al.*, 1997; Kneer and Law, 1998; Law and Eberly, 1996; Monroe *et al.*, 1996; Steinbach *et al.*, 1997; Vogel and de Matos Filho, 1995; Wineland *et al.*, 1998). Thus the dynamics is reduced to that of the motional degrees of freedom only. Under this Hamiltonian the motional dynamics evolves toward the DFSs $|\alpha, m\rangle$ when it is prepared initially in the Fock state $|m\rangle$. The state $|m\rangle$ can be prepared with a very high efficiency according to recent experiments (de Matos Filho and Vogel, 1996; Gardiner *et al.*, 1997; Itano *et al.*, 1997; Kneer and Law, 1998; Law and Eberly, 1996; Monroe *et al.*, 1996; Steinbach *et al.*, 1997; Vogel and de Matos Filho, 1995; Wineland *et al.*, 1998). The preparation of superposition of these states can be done according to the scheme described here.

We start from

$$|\Psi(0)\rangle = \sum_{n=0}^{m} c_n |n, g\rangle. \tag{2.6}$$

This state can be generated by successive applications of an external classical driving field and a quantized field as described in detail in previous studies (Abd Al-Kader, 1999; de Matos Filho and Vogel, 1996; Gardiner *et al.*, 1997; Itano *et al.*, 1997; Kneer and Law, 1998; Law and Eberly, 1996; Monroe *et al.*, 1996; Obada and Abd Al-Kader, 1998, 1999; Schleich and Raymer, 1997; Steinbach *et al.*, 1997; Vogel and de Matos Filho, 1995; Wineland *et al.*, 1998). Applying classical field (carrier) for a duration time τ_1 whose evolution operator takes the form

$$\begin{aligned} U_1(\tau_1) = {} & \cos\Omega_1\tau_1|e\rangle\langle e| - i\, e^{e^{i\theta_1}} \sin\Omega_1\tau_1|e\rangle\langle g| \\ & - i\, e^{-i\theta_1} \sin\Omega_1\tau_1|g\rangle\langle e| + \cos\Omega_1\tau_1|g\rangle\langle g|, \end{aligned} \tag{2.7}$$

where Ω_1 is the Rabi frequency in this case and θ_1 is a phase on the state (2.6), and taking $\Omega_1\tau_1 = \pi/4,\ \theta_1 = \pi/2$, we get

$$|\xi(\tau_1)\rangle = U_1(\tau_1)|\xi(0)\rangle = \sum_{n=0}^{m} \frac{c_n}{\sqrt{2}}|n\rangle \otimes (|e\rangle + |g\rangle). \tag{2.8}$$

The internal state $(|e\rangle + |g\rangle)$ will remain constant under the Hamiltonian (2.4). Applying the Hamiltonian $H_I^{(1)}$ for a time duration τ_2. The state $|\xi(\tau_1)\rangle$ evolves to

$$|\xi(\tau_2)\rangle = |\xi(\tau_2 + \tau_1)\rangle = U_1(\tau_2)|\xi(\tau_1)\rangle = \sum_{n=0}^{m} \frac{c_n}{\sqrt{2}}|\alpha, n\rangle \otimes (|e\rangle + |g\rangle), \tag{2.9}$$

where $\alpha = 2i(g_1 + g_2^*)\tau_2$. The state in (2.9) is a superposition of displaced Fock states but with the same displacement α. This state is equivalent to that discussed in Obada and Abd Al-Kader (1998, 1999) and Abd Al-Kader (1999).

We choose the polarization in the quantized field so that it affects the excited state only as described in previous studies (de Matos Filho and Vogel, 1996; Gardiner *et al.*, 1997; Itano *et al.*, 1997; Kneer and Law, 1998; Law and Eberly, 1996; Monroe *et al.*, 1996; Steinbach *et al.*, 1997; Vogel and de Matos Filho, 1995; Wineland *et al.*, 1998) and apply the linear field in the Hamiltonian (i.e., $H_I^{(1)}$) for a duration τ_3, which generates the state

$$|\xi(\tau_3)\rangle = \acute{U}_2(\tau_3)|\xi(\tau_2)\rangle = \sum_{n=0}^{m} \frac{c_n}{\sqrt{2}}[|\beta, n\rangle|e\rangle + |\alpha, n\rangle|g\rangle], \tag{2.10}$$

where $\beta = \alpha + 2i(\acute{g}_1 + \acute{g}_2^*)\tau_3$.

After that we apply a carrier pulse for a duration τ_4 with the evolution operator (2.7). It produces the following state

$$\begin{aligned}|\xi(\tau_4)\rangle &= \sum_{n=0}^{m} \frac{c_n}{\sqrt{2}}[|\beta, n\rangle(\cos\Omega_1\tau_4|e\rangle - i\, e^{-i\theta_2}\sin\Omega_1\tau_4|g\rangle) \\ &\quad + |\alpha, n\rangle(-i\, e^{i\theta_2}\sin\Omega_1\tau_4|e\rangle + \cos\Omega_1\tau_4|g\rangle)] \\ &= \sum_{n=0}^{m} \acute{c}_n[(\cos\Omega_1\tau_4|\beta, n\rangle - i\, e^{i\theta_2}\sin\Omega_1\tau_4|\alpha, n\rangle)|e\rangle) \\ &\quad + (\cos\Omega_1\tau_4|\alpha, n\rangle - i\, e^{-i\theta_2}\sin\Omega_1\tau_4|\beta, n\rangle)|g\rangle] \\ &= \sum\{(C_{1n}|\beta, n\rangle + C_{2n}|\alpha, n\rangle)|e\rangle + (D_{1n}|\beta, n\rangle \\ &\quad + D_{2n}|\alpha, n\rangle)|g\rangle\}. \end{aligned} \tag{2.11}$$

Apply the linear field for a duration τ_5 with the polarization chosen, so that it generates the state

$$\begin{aligned}|\xi(\tau_5)\rangle &= \acute{U}_2(\tau_5)|\xi(\tau_4)\rangle \\ &= \sum\{(C_{1n}|\beta_1, n\rangle + C_{2n}|\alpha_1, n\rangle)|e\rangle \\ &\quad + (D_{1n}|\beta, n\rangle + D_{2n}|\alpha, n\rangle)|g\rangle\}. \end{aligned} \tag{2.12}$$

After that we apply a carrier pulse for a duration τ_7 with the evolution operator (2.7). We have

$$\begin{aligned}|\xi(\tau_6)\rangle &= U_1(\tau_6)|\xi(\tau_5)\rangle \\ &= \sum\{[\acute{C}_{1n}|\beta_1, n\rangle + \acute{C}_{2n}|\alpha_1, n\rangle + \acute{D}_{1n}|\beta, n\rangle + \acute{D}_{2n}|\alpha, n\rangle]|e\rangle \\ &\quad + [\acute{K}_{1n}|\beta_1, n\rangle + \acute{K}_{2n}|\alpha_1, n\rangle + (\acute{B}_{1n}|\beta, n\rangle + \acute{B}_{2n}|\alpha, n\rangle]|g\rangle\}. \end{aligned} \tag{2.13}$$

Detecting the atom in either of its electronic states gives

$$|\Psi\rangle = \sum_{n=0}^{m}[A_n|\beta, n\rangle + B_n|\alpha, n\rangle + C_n|\beta_1, n\rangle + K_n|\alpha_1, n\rangle]. \tag{2.14}$$

The states (2.14) correspond to a system of four-component superposition of the DFSs. After the proposal scheme has been made, we will examine the possible occurrence of nonclassical effects exhibited by the states of Eq. (2.14).

3. CHARACTERISTIC FUNCTION OF FOUR-COMPONENT OF DFSs

It is well-known that the DFSs is defined by (Abd Al-Kader, 1994; Agarwal and Tara, 1991; Boiteux and Levelut, 1973; De Oliveira *et al.*, 1990; Moya-Cessa and Knight, 1993; Roy and Singh, 1982; Satyanarayana, 1985; Wunsche, 1991)

$$|\alpha, m\rangle = D(\alpha)|m\rangle = \sum_{n=0}^{\infty} a(n, m)|n\rangle. \tag{3.1}$$

The operator $D(\alpha) = \exp(\alpha a^{+} - \alpha^{*} a)$ and $\alpha = |\alpha|e^{i\theta}$ is the displacement operator (Glauber, 1963; Perina, 1984; Walls and Milburn, 1994), where $a(a^{+})$ is the annihilation (creation) operator of the boson field.

The scalar product $\langle\beta, m \mid \alpha, n\rangle$ is given by (Abd Al-Kader; 1994; Agarwal and Tara, 1991; De Oliveira *et al.*, 1990; Moya-Cessa and Knight, 1993; Wunsche, 1991)

$$\langle\beta, m \mid \alpha, n\rangle = \begin{cases} \langle\beta \mid \alpha\rangle\sqrt{\frac{n!}{m!}}(\alpha - \beta)^{m-n} L_{n}^{m-n}(|\alpha - \beta|^{2}), & m > n \\ \langle\beta \mid \alpha\rangle\sqrt{\frac{m!}{n!}}(\beta^{*} - \alpha^{*})^{n-m} L_{m}^{n-m}(|\alpha - \beta|^{2}), & n > m \end{cases} \tag{3.2}$$

where the scalar product of two coherent states has the well-known value $\langle\beta \mid \alpha\rangle = \exp[(-1/2)(|\alpha|^{2} + |\beta|^{2}) + \alpha\beta^{*}]$, and $L_{m}^{\sigma}(x)$ is the Laguerre polynomial

$$L_{m}^{\sigma}(x) = \sum_{s=0}^{m} \binom{m+\sigma}{m-s} \frac{(-x)^{s}}{s!}. \tag{3.3}$$

Let the wider class of quantum state $|\Psi_{N}\rangle$ have the form

$$|\Psi_{N}\rangle = A_{N}^{-\frac{1}{2}} \sum_{j=1}^{N} k_{j} |\alpha_{j}, m_{j}\rangle, \tag{3.4}$$

with A_{N} as the normalization constant.

To shed some light on this state (3.4) we shall be more specific and consider the superposition of four DFSs in the form

$$\begin{aligned} |\Psi_{4}\rangle = A_{4}^{-\frac{1}{2}}[&|\alpha_{0}, m\rangle + \exp(i\xi_{1})|-\alpha_{0}, m\rangle \\ &+ \exp(i\xi_{2})|i\alpha_{0}, m\rangle + \exp(i\xi_{3})|-i\alpha_{0}, m\rangle], \end{aligned} \tag{3.5}$$

where the normalization constant A_{4} is given by

$$\begin{aligned} A_{4} = \big\{&4 + 2\cos\xi_{1} e^{-2|\alpha|^{2}} L_{m}(4|\alpha_{0}|^{2}) + 2\cos(\xi_{3} - \xi_{2}) e^{-2|\alpha|^{2}} L_{m}(4|\alpha_{0}|^{2}) \\ &+ 2[\cos(|\alpha_{0}|^{2} + \xi_{2}) + \cos(|\alpha_{0}|^{2} + \xi_{3} - \xi_{1})] e^{-|\alpha_{0}|^{2}} L_{m}(|2\alpha_{0}|^{2}) \\ &+ 2[\cos(|\alpha_{0}|^{2} + \xi_{3}) + \cos(|\alpha_{0}|^{2} + \xi_{1} - \xi_{2})] e^{-1/2|\alpha_{0}|^{2}} L_{m}(|2\alpha_{0}|^{2})\big\}. \end{aligned} \tag{3.6}$$

The s-parameterized CF is perhaps one of the most well-known functions in quantum optics, since it is the Fourier transformation of the s-parameterized QDF.

The s-parameterized CF is defined by (Cahill and Glauber, 1969; Glauber, 1963; Perina, 1984; Walls and Milburn, 1994)

$$C(\lambda, s) = \mathrm{Tr}[\rho D(\lambda)] \exp\left(\frac{s}{2}|\lambda|^2\right), \tag{3.7}$$

with $D(\lambda)$ as given before (see after Eq. (3.1)). Here, s is an ordering parameter where $s = (-1)\,1$ means (anti-)normal ordering and $s = 0$ is symmetrical or Weyl ordering (Cahill and Glauber, 1969; Wünsche, 1998). Using the density operator $\rho = |\Psi_4\rangle\langle\Psi_4|$, we find the corresponding s-parameterized CF in the form

$$\begin{aligned}
C_4(\lambda, s) = {} & A_4^{-1}\exp\left(\frac{s}{2}|\lambda|^2\right)\left\{\left[\exp\left\{-\frac{1}{2}|\lambda|^2\right\}L_m(|\lambda|^2)\right][\exp(\alpha_0^*\lambda - \alpha_0\lambda^*)\right. \\
& + \exp(-\alpha_0^*\lambda + \alpha_0\lambda^*) + \exp(-i\alpha_0^*\lambda - i\alpha_0\lambda^*) + \exp(i\alpha_0^*\lambda + i\alpha_0\lambda^*)] \\
& + \exp\left\{i\xi_1 - \frac{1}{2}|\lambda - 2\alpha_0|^2\right\}L_m(|\lambda - 2\alpha_0|^2) \\
& + \exp\left\{-i\xi_1 - \frac{1}{2}|\lambda + 2\alpha_0|^2\right\}L_m(|\lambda + 2\alpha_0|^2) \\
& + \exp\left\{i(\xi_3 - \xi_2) - \frac{1}{2}|\lambda - 2i\alpha_0|^2\right\}L_m(|\lambda - 2i\alpha_0|^2) \\
& + \exp\left\{-i(\xi_3 - \xi_2) - \frac{1}{2}|\lambda + 2i\alpha_0|^2\right\}L_m(|\lambda + 2i\alpha_0|^2) \\
& + \exp\left\{-\frac{1}{2}|\lambda + i\alpha_0 - \alpha_0|^2\right\}L_m(|\lambda + i\alpha_0 - \alpha_0|^2) \\
& \times\left[\exp\left(i\xi_2 + \frac{1}{2}\{(-i+1)\alpha_0^*\lambda - (i+1)\alpha_0\lambda^*\} + i|\alpha_0|^2\right)\right. \\
& \left. + \exp\left(i(\xi_1 - \xi_3) + \frac{1}{2}\{(i-1)\alpha_0^*\lambda + (i+1)\alpha_0\lambda^*\} - i|\alpha_0|^2\right)\right] \\
& + \exp\left\{-\frac{1}{2}|\lambda - i\alpha_0 - \alpha_0|^2\right\}L_m(|\lambda - i\alpha_0 - \alpha_0|^2) \\
& \times\left[\exp\left(i\xi_3 + \frac{1}{2}\{(i+1)\alpha_0^*\lambda + (i-1)\alpha_0\lambda^*\} - i|\alpha_0|^2\right)\right. \\
& \left. + \exp\left(i(\xi_1 - \xi_2) + \frac{1}{2}\{-(i+1)\alpha_0^*\lambda - (i-1)\alpha_0\lambda^*\} + i|\alpha_0|^2\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\exp\left\{-\frac{1}{2}|\lambda + i\alpha_0 + \alpha_0|^2\right\}L_m(|\lambda + i\alpha_0 + \alpha_0|^2)\\
&\times\left[\exp\left(i(\xi_3 - \xi_1) + \frac{1}{2}\{-(i+1)\alpha_0^*\lambda - (i-1)\alpha_0\lambda^*\} - i|\alpha_0|^2\right)\right.\\
&\left.+\exp\left(-i\xi_3 + \frac{1}{2}\{(i+1)\alpha_0^*\lambda + (i-1)\alpha_0\lambda^*\} + i|\alpha_0|^2\right)\right]\\
&+\exp\left\{-\frac{1}{2}|\lambda - i\alpha_0 + \alpha_0|^2\right\}L_m(|\lambda - i\alpha_0 + \alpha_0|^2)\\
&\times\left[\exp\left(i(\xi_3 - \xi_1) + \frac{1}{2}\{(i-1)\alpha_0^*\lambda + (i+1)\alpha_0\lambda^*\} + i|\alpha_0|^2\right)\right.\\
&\left.\left.+\exp\left(-i\xi_3 + \frac{1}{2}\{-(i-1)\alpha_0^*\lambda - (i+1)\alpha_0\lambda^*\} - i|\alpha_0|^2\right)\right]\right\}. \qquad (3.8)
\end{aligned}
$$

Once the s-parameterized CF is obtained, one can calculate any expectation value for the field operators from it.

4. THE NONCLASSICAL PROPERTIES OF FOUR-COMPONENT DFSs

We now examine these states for specific nonclassical properties. There are essentially two possible ways for nonclassical effects to manifest themselves: One is the occurrence of sub-Poissonian statistics (amplitude squeezing), the second is quadrature squeezing.

4.1. Photon Number Distribution

We begin by looking at the photon number distribution for the state $|\Psi_4\rangle$. The photon number normal generating function $C^{(N)}(\lambda, 1)$ is simply connected to the normal ($s = 1$) CF by the relation (Glauber, 1963; Perina, 1984; Walls and Milburn, 1994)

$$C^{(N)}(\lambda, 1) = \frac{1}{\pi\lambda}\int \exp\left(-\frac{|\beta|^2}{\lambda}\right)C(\beta, 1)\, d^2\beta, \qquad (4.1)$$

and it generates photon number distributions in the form

$$P(n) = \frac{(-1)^n}{n!}\frac{d^n}{d\lambda^n}C^{(N)}(\lambda, 1)|_{\lambda=1}. \qquad (4.2)$$

Also, the photon number distribution $P(n)$ can be obtained through the relation

$$P(n) = |\langle n \mid \Psi_4\rangle|^2. \qquad (4.3)$$

In Fig. 1 we illustrate the photon number distribution $P(n)$ with $\alpha_0 = 4$, $\xi_i = 0$, $i = 1, 2, 3$. The excited photon number m is assumed as (A) $m = 1$ and (B) $m = 2$. These figures show the oscillations of the distribution that result from interference between the different DFSs. It is found that $P(4n + 1)$ are nonzero for $m = 1$ while $P(4n + 2)$ are nonzero for $m = 2$, as could be expected.

4.2. Autocorrelation Function $g^{(2)}(0)$

To characterize the width of the distribution, it is convenient to use the auto-correlation function defined by

$$g^{(2)}(0) = \frac{\langle a^{+2}a^2\rangle}{\langle a^+a\rangle^2} = 1 + \frac{\langle \Delta n^2\rangle - \langle n\rangle}{\langle n\rangle^2}, \tag{4.4}$$

where $n = a^+a$ is the photon number operator. Whenever $0 \leq g^{(2)}(0) < 1$, or whenever $\langle \Delta n^2\rangle < \langle n\rangle$, photon antibunching is said to exist.

The function $g^{(2)}(0)$ has been classified such that the light with $g^{(2)} < 1$ has a sub-Poissonian distribution, the light with $1 < g^{(2)} < 2$ has a super-Poissonian distribution, and the light with $g^{(2)} > 2$ is called super thermal light. It is well known that the coherency is unity for coherent light (with Poissonian distribution).

One readily finds the expectation values

$$\begin{aligned}\langle [a^{+^k} a^l]_s\rangle &= \mathrm{Tr}\left[\rho\left\{a^{+^k} a^l\right\}_s\right] \\ &= \frac{\partial^k}{\partial\lambda^k}\frac{\partial^l}{\partial(-\lambda^*)^l}C(\lambda, s)\mid_{\lambda=\lambda^*=0}\end{aligned} \tag{4.5}$$

that are necessary to examine statistical properties of the field state. In addition, it is used to calculate the correlation function $g^{(2)}(0)$.

In Fig. 2 we plot the auto-correlation function $g^{(2)}(0)$ against the displacement parameter α_0, with $\xi_i = 0$, $i = 1, 2, 3$ and $m = 1, 2, 3, 5$. It is apparent that the light is sub-Poissonian light for $\alpha_0 < 1$ for the values of m considered. But it becomes Poissonian for increasing values of α_0.

4.3. Squeezing

We next look at squeezing. To do this, we introduce the two quadrature operators

$$X_1 = \frac{1}{2}(a + a^+) \quad \text{and} \quad X_2 = \frac{1}{2i}(a - a^+). \tag{4.6}$$

These are dimensionless position and momentum operators for a harmonic oscillator. They satisfy $[X_1, X_2] = i/2$. The uncertainty relation in this case is $\langle(\Delta X_1)^2\rangle\langle(\Delta X_2)^2\rangle \geq 1/16$ with the variance $\langle(\Delta X_j)^2\rangle = \langle X_j^2\rangle - \langle X_j\rangle^2$. The field is said to be squeezed if $(\Delta X_j)^2 < 1/4$ for $j = 1$ or 2.

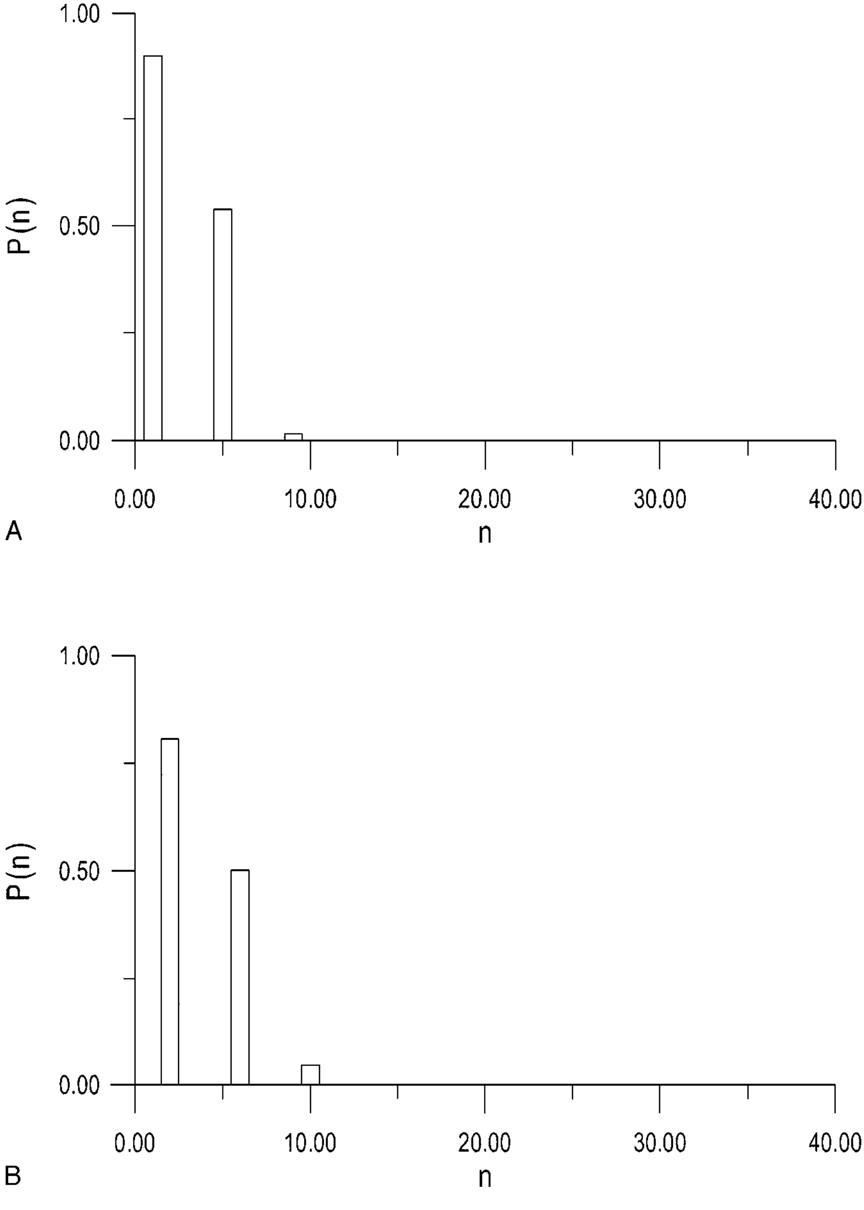


Fig. 1. We illustrate the photon number distribution $P(n)$ with $\alpha_0 = 4$, $\xi_i = 0$, $i = 1, 2, 3$, and $m = 1, 2$ in Fig. 1(A) and (B) respectively.

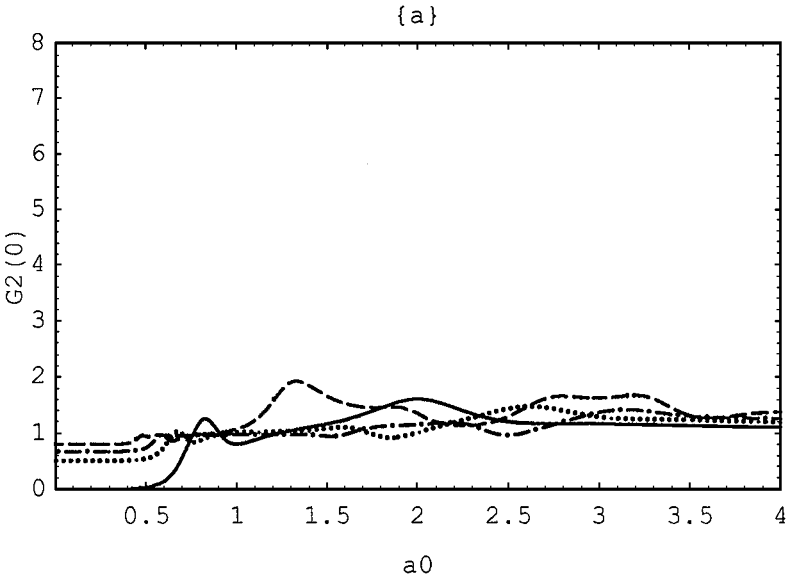


Fig. 2. Coherence function $g^{(2)}$ measured on vertical axis and horizontal axis indicates the displacement parameter α_0. The number of photons of initial state (Four-component superposition of DFSs) have the values $m = 1$ (solid curve), $m = 2$ (dotted curve), $m = 3$ (chained curve) and $m = 5$ (dashed curve). The remainder parameters assume the same values in Fig. 1.

The average values of the quadrature field operators $\langle X_1 \rangle$ and $\langle X_2 \rangle$ are directly computed. Also, variances of the quadrature field operators $\langle (\Delta X_1)^2 \rangle$ and $\langle (\Delta X_2)^2 \rangle$ are calculated.

The squeezing is best parameterized by

$$q_j = \frac{\langle (\Delta X_j)^2 \rangle - 0.25}{0.25}, \quad j = 1, 2, \tag{4.7}$$

such that squeezing exits for $-1 < q_j < 0$, i.e., the squeezing condition now reads $q_j < 0$, and the maximum squeezing corresponds to $q_j = -1$. Squeezing in one quadrature is achieved at the expense of increased noise in the conjugate quadrature; therefore, if one of the q_js is less than zero, the other should be greater than zero.

The quadratures have no squeezing for these states with the same parameters of Fig. 2.

4.4. Phase Distribution

In the Barnett and Pegg (1986, 1989a,b) phase operator formalism, all physical quantities are calculated in $(s+1)$-dimensional space. After all calculations are completed, s is made infinite. the Pegg–Barnett phase operator and its eigenstates are defined in $(s+1)$-dimensional space Ψ spanned by the number states $|0\rangle, |1\rangle, \ldots, |s\rangle$ Once the phase distribution function $P(\theta)$ is known, all the quantum mechanical phase expectation values can be calculated with this function in a manner similar to the classical one by integration over θ ("Quantum phase and phase dependent measurements," 1993; Lynch, 1995; Perinova *et al.*, 1998).

The Pegg–Barnett phase distribution is given by

$$P(\theta) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n C_m^* \exp[-i(n-m)\theta], \tag{4.8}$$

or we can write it in the form

$$P(\theta) = \frac{1}{2\pi} \left\{ 1 + 2Re \sum_{n>m}^{\infty} C_n C_m^* \exp[-i(n-m)\theta] \right\} \tag{4.9}$$

for the state $|\Psi_4\rangle$ defined in terms of number (Fock) states on the form

$$|\Psi_4\rangle = \sum_{n=0}^{\infty} C_n |n\rangle, \tag{4.10}$$

where $C_n = \langle n \mid \Psi_4\rangle$ is the amplitude.

In Fig. 3 we plot the phase distribution $P(\theta)$ for $|\alpha_0| = 1$, with $\xi_i = 0$, $i = 1, 2, 3$, and $m = 1$ (solid curve) and $m = 2$ (dotted curve). For small values of the displacement parameter α_0 there is no information about the phase. This can be atributed to the effect of the number state $|m\rangle$. The four peaks is found for the phase distribution with increasing amplitude α_0. Calculations show that the same four peaks are symmetrically located at $\pm\pi/2$, 0, and the two wings at $\pm\pi$, but with large amplitude for increased α_0.

4.5. Quadrature Distributions

To calculate the quadrature component distribution for the superposition state (i.e., the phase-parameterized field strength distribution) we write

$$P(x, \Phi) = |\langle x, \Phi \mid \Psi_4\rangle|^2, \tag{4.11}$$

which can be measured in balanced homodyne detection Dakna *et al.* (1997, 1998). We first expand the eigenstate $|x, \Phi\rangle$ of the quadrature component

$$x(\Phi) = \frac{1}{\sqrt{2}}(e^{-i\Phi}a + e^{i\Phi}a^+) \tag{4.12}$$

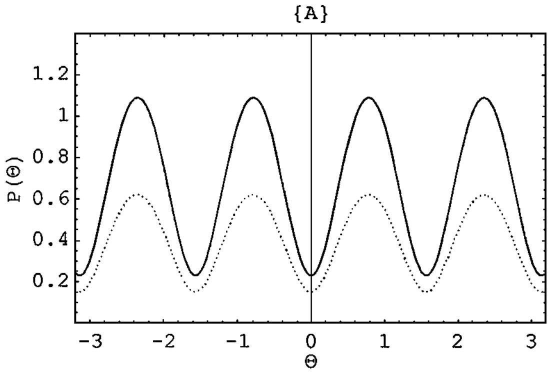


Fig. 3. The phase distribution for the displacement parameter $\alpha_0 = 1$, with $m = 1$ (solid curve) and $m = 2$ (dotted curve). The remainder parameters assume the same values as in Fig. 1.

in the photon number basis as Dakna *et al.* (1997, 1998)

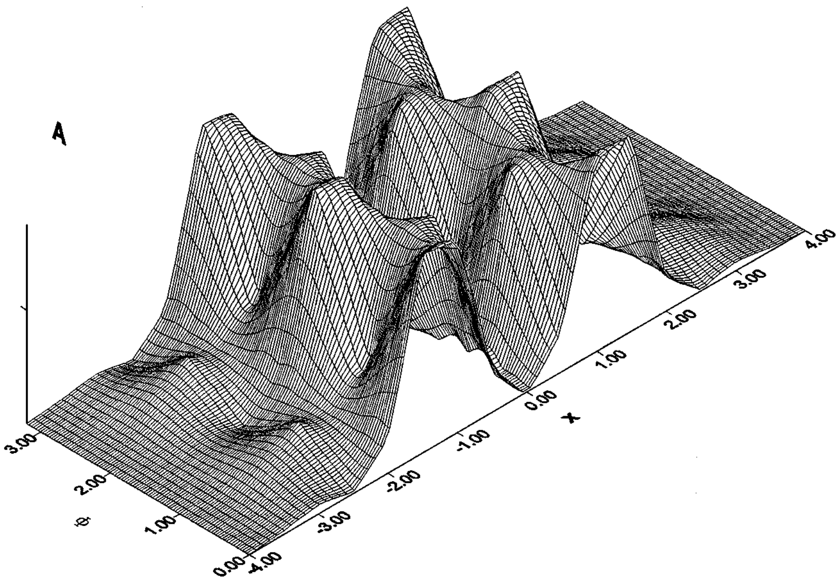
$$|x, \Phi\rangle = \frac{1}{\pi^{\frac{1}{4}}} \exp\left(-\frac{1}{2}x^2\right) \sum_{j=0}^{\infty} \frac{e^{i\Phi j}}{\sqrt{2^j j!}} H_j(x)|j\rangle. \tag{4.13}$$

By using Eqs. (4.10) and (4.13) we have the quadrature component distribution (4.11) in the form

$$P(x, \Phi) = \frac{1}{A\pi^{\frac{1}{2}}} \exp(-x^2) \sum_{j,l=0}^{\infty} \frac{\exp[\Phi(l - j)]}{\sqrt{2^{(l+j)}} j!l!} C_j C_l^* H_j(x) H_l(x). \tag{4.14}$$

In Fig. 4 we plot the phase-parameterized field strength distribution (quadrature component) $P(x, \Phi)$ with $\alpha_0 = 1$. The excited photon number is assumed as (A) $m = 1$ and (B) $m = 2$. In general the figure for $P(x, \Phi)$ is symmetric around $x = 0$ and $\Phi = \pi/2$. Changing the displacement parameter α_0 makes a marked difference for the distributions. For $m = 1$, the two-peak shape is clear for $\Phi = 0$; for $m = 2$, the three-peak shape is shown for $\phi = 0$.

For small values of α_0, it is observed that one-peak shape for the distribution for $\Phi = 0$ or π disappears as Φ increases and diverges as Φ gets closer to $\pi/2$. However, numerical calculations show that increasing the parameter α_0 adds further oscillations to the quadrature distribution $P(x, \Phi)$. In addition, the interference can exist at large values of α_0. The number of peaks increases with the increase of m.



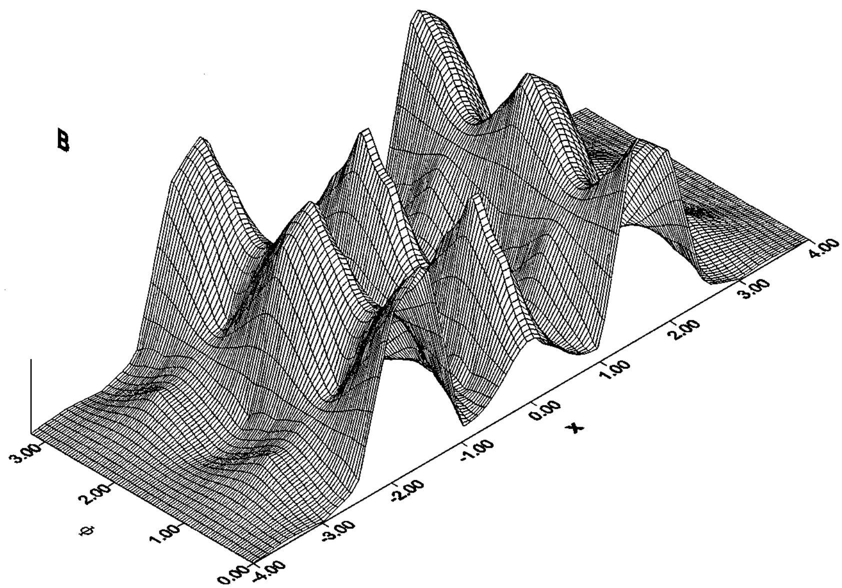


Fig. 4. Plots of the phase-parameterized field strength distribution (quadrature component) $P(x,\ \Phi)$, with $\alpha_0 = 1$, $\xi_i = 0$, $i = 1, 2, 3$ and (A) $m = 1$, (B) $m = 2$.

4.6. Quasiprobability Functions

The computation of quasiprobability functions, given a density matrix, is often a tedious task that involves integration over phase space variables. The exception is the Q function, which is simply expressed as the coherent expectation value of the field density matrix and is therefore widely adopted to describe field dynamics in situations where the density matrix is easily computed. However, the Wigner function has an interesting characteristic that makes it an excellent diagnostic of quantum properties. It has negative values in some areas of the phase space for the nonclassical field states (Hillery *et al.*, 1984; Lee, 1995).

Here we examine the s-parameterized quasiprobability function associated with our states. As is well known these s-parameterized quasiprobability functions provide a way to characterize the nonclassical nature of a quantum field. The s-parameterized quasi-probability function is the Fourier transformation of the s-parameterized characteristic function

$$F(\beta, s) = \frac{1}{\pi^2} \int C(\lambda, s)\ \exp(\lambda^* \beta - \lambda \beta^*)\, d^2\lambda \tag{4.15}$$

where the real parameter s defines the corresponding phase space distribution. It is well known that such a parameter is associated with the ordering of the field bosonic operators. For example, $s = 1$, 0, and -1 correspond to the normal, symmetric, and antinormal ordering, respectively. The corresponding quasiprobability functions are the P function, the Wigner function, and Q function. The general expression of s-parameterized characteristic function in Eq. (3.8) may be employed to facilitate the evaluation of the s-parameterized quasiprobability function. The problems in this scheme arise in the integration of Eq. (4.15), which is not easy to calculate, so that we consider the Q function only, which is defined by (Glauber, 1963; Perina, 1984; Walls and Milburn, 1994)

$$Q(\beta) = \frac{1}{\pi} |\langle \beta \mid \Psi_4 \rangle|^2 \tag{4.16}$$

where $|\beta\rangle$ is a coherent state. More than just a theoretical curiosity, $Q(\beta)$ can be detected in homodyne experiments (Leonhardt, 1997; Leonhardt and Paul, 1995). This distribution function $Q(\beta)$ has no singularity problems at all. It exists for all density matrixes, is bounded, and is even greater than or equal to zero for all β. It has the form

$$Q(\beta) = \frac{1}{\pi}\ \exp(-|\beta|^2) \sum_{n,m=0}^{\infty} C_n C_m^* \frac{(\beta^*)^n (\beta)^m}{\sqrt{n!m!}} \tag{4.17}$$

In Fig. 5 we have sketched the Q function with $\alpha_0 = 4$ and $m = 1$. We show that the four sets of hollowed-pek structures of DFSs is observed. As α_0 increases the four sets of peaks are seperated away from the center $x = \Re\beta = 0$, $y = \Im\beta = 0$.

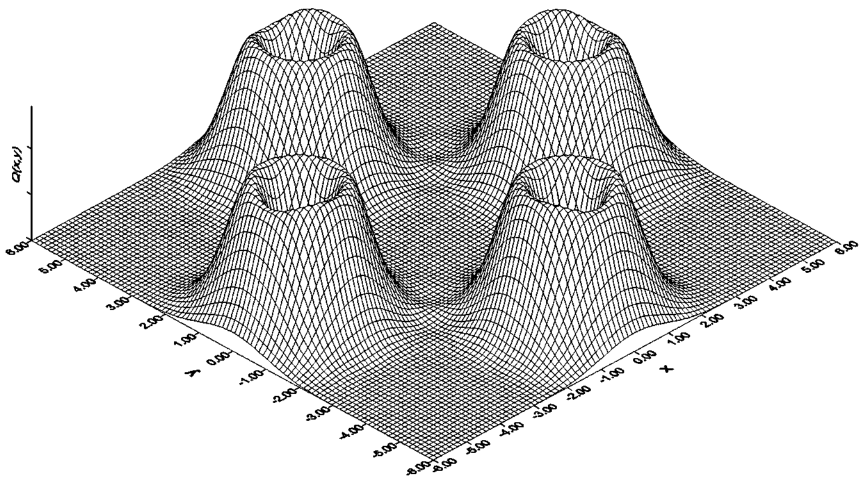


Fig. 5. The Q function for the four-component DFSs superposition states. The parameters are assumed as $\alpha_0 = 4$ and $m = 1$. The remainder parameters assume the same values as in Fig. 1. Here $x = \Re(\beta)$ and $y = \Im(\beta)$.

The Wigner function is usually expressed in an integral form, which is not always easy to compute as shown here. Recently Wünsche (1998) has derived another form for the Wigner function, and in general, for s-parameterized quasiprobability function (Cahill and Glauber, 1969; Glauber, 1963; Perina, 1984; Walls and Milburn, 1994; Wünsche, 1998). According to Wünsche (1998) the Wigner function is given by

$$W(\beta) = \frac{2}{\pi}\ \exp(-2|\beta|^2) \sum_{n,m=0}^{\infty} C_n C_m^*(-1)^n \sqrt{\frac{n!}{m!}}(2\beta^*)^{m-n} L_n^{m-n}(4|\beta)|^2) \quad (4.18)$$

Fig. 6 shows plots for the Wigner function for the same parameters of Fig. 5. It is evident that the function takes on negative values over some range of x and y, thus indicating the nonclassical nature of the states. Calculations show that the function takes on more negative values for increasing m. It is seen that oscillatory behavior of the Wigner function of four-component DFSs is observed.

5. CONCLUSIONS

We have discussed the properties and a generation scheme of four-component DFSs. A generation scheme for these states has been presented. This scheme depends on driving the vibrational motion of a trapped ion to any quantum state based on sequence of excitations of the ion by a classical laser field. During each excitation, the laser field is turned to the respective lower vibrational sideband. Therefore,

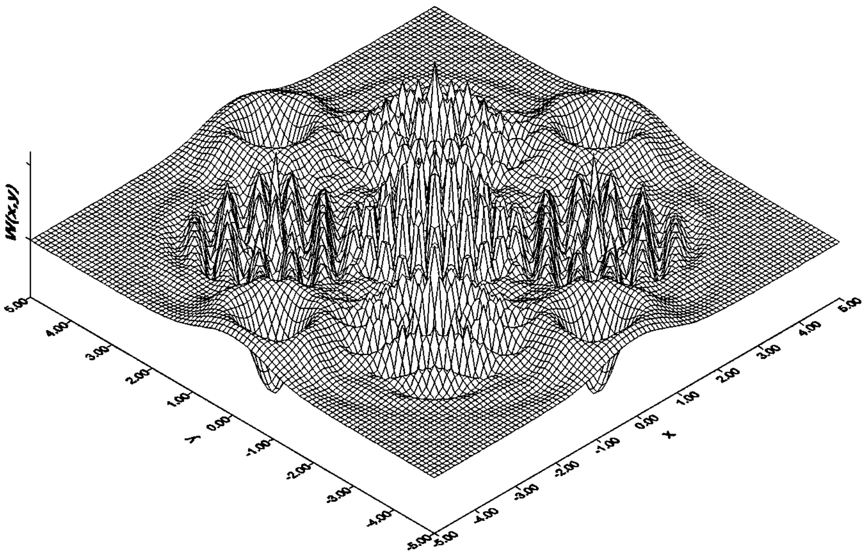


Fig. 6. The Wigner function for the four-component DFSs superposition states. The parameters are assumed as in Fig. 5.

the vibrational motion is prepared in the desired state of a four-component superposition of DFSs. Based on the currently available techniques [Wineland group, (de Matos Filho and Vogel, 1996; Gardiner *et al.*, 1997; Itano *et al.*, 1997; Kneer and Law, 1998; Law and Eberly, 1996; Monroe *et al.*, 1996; Steinbach *et al.*, 1997; Vogel and de Matos Filho, 1995; Wineland *et al.*, 1998)], the scheme may be realizable.

We have discussed the photon number distribution, *s*-parameterized characteristic function, and quasiprobability distribution function. The three-dimensional plots of the Wigner and Q functions for some parameters have been illustrated. Several moments have been calculated by using the characteristic function. The second-order correlation function $g^{(2)}(0)$ has been investigated numerically. The squeezing properties for these states have been discussed. We have analyzed the quadrature component distributions for the superposition of four DFSs and have presented analytical and numerical results. We have found that the basic features of superposition of a four DFS, such as the appearance of several separated peaks and an interference pattern, are present.

The present work was motivated by the desire to realize physically certain specific quantum states (superposition of DFSs). It is hoped that the superposition of DFSs will find application in quantum computer, quantum information Shor and Preskill (2000), and quantum optics.

REFERENCES

Abd Al-Kader, G. M. (1994). *Journal of the Egyptian Mathematical Society* **2**, 89.
Abd Al-Kader, G. M. (1999). *Pramana—Journal Physics* **53**, 877.
Abdalla, M. S., Mahran, M. H., and Obada, A.-S. F. (1994). *Journal of Modern Optics* **41**, 1889.
Agarwal, G. S. and Tara, K. (1991). *Physical Review A* **43**, 492.
Ban, M. (1995). *Physical Review A* **51**, 1604.
Barnett, S. M. and Pegg, D. T. (1986). *Journal of Physics A* **19**, 3849.
Barnett, S. M. and Pegg, D. T. (1989a). *Physical Review A* **39**, 1665.
Barnett, S. M. and Pegg, D. T. (1989b). *Journal of Modern Optics* **36**, 7.
Boiteux, M. and Levelut, A. (1973). *Journal of Physics A: Mathemtical and General* **6**, 589.
Bose, S., Jacabs, K., and Knight, P. L. (1997). *Physical Review A* **56**, 4175.
Buzek, V., Vidiella-Barranco, A., and Knight, P. L. (1992). *Physical Review A* **45**, 6570.
Cahill, K. E. and Glauber, R. J. (1969). *Physical Review* **177**, 1857, 1882.
Carruthers, P. and Nieto, M. M. (1968). *Reviews of Modern Physics* **40**, 411.
Dakna, M., Anhut, T., Opatrny, T., Knoll, L., and Welsch, O.-G. (1997). *Physical Review A* **55**, 3184.
Dakna, M., Knoll, L., and Welsch, D.-G. (1998). *European Physics Journal D* **3**, 295.
de Matos Filho, R. L. and Vogel, W. (1996). *Physics Review Letters* **76**, 608.
De Oliveira, F. A. M., Kim, M. S., Knight, P. L., and Buzek, V. (1990). *Physical Review A* **41**, 2645.
Dodonov, V. V., Korennoy, Y. A., Man'ko, V. I., and Moukhin, Y. A. (1996). *Quantum Semiclassical Optics* **8**, 413.
Dodonov, V. V., Marciolli, M. A., Korennoy, Y. A., Man'ko, V. I., and Moukhin, Y. A. (1998). *Physical Review A* **58**, 4087.
Gardiner, S. A., Cirac, J. I., and Zoller, P. (1997). *Physical Review A* **55**, 1683.
Garraway, B. M., Knight, P. L., and Steinbach (1995). *Journal of Applied Physics B* **60**, s63.
Garraway, B. M., Sherman, H., Moya-Cessa, H., Knight, P. L., and Kurizki, G. (1994). *Physical Review A* **49**, 535.
Glauber, R. J. (1963). *Physical Review* **131**, 2766.
Gou, S.-C., Steinbach, J., and Knight, P. L. (1997). *Physical Review A* **55**, 3719.
Hach, E. E., III and Gerry, C. C. (1992). *Journal of Modern Optics* **39**, 2501.
Hach, E. E., III and Gerry, C. C. (1993). *Journal of Modern Optics* **40**, 2351.
Hillery, M., O'Connell, R. F., Scully, M. O., and Wigner, E. P. (1984). *Physical Reports* **106**, 121.
Itano, W. M., Monroe, C., Meekhof, D. M., Leifried, D., King, B. E., and Wineland, D. J. (1997). SPIE Proceedings **2995**, 43.
Jackiw, R. J. (1968). *Mathemetical Physics* **9**, 339.
Kneer, B. and Law, C. K. (1998). *Physical Review A* **57**, 2096.
Law, C. K. and Eberly, J. H. (1996). *Physical Review Letters* **76**, 1055.
Lee, H.-W. (1995). *Physics Reports* **259**, 147.
Lee, K. S., Kim, M. S., Lee, S.-D., and Buzek, V. (1993). *Journal of the Korean Physical Society A* **26**, 197.
Leonhardt, U. and Paul, H. (1995). *Progress Quantum Electronics* **19**, 89.
Leonhardt, U. (1997). *Measuring the Quantum State of Light*, Cambridge University Press, Cambridge.
Loudon, R. (1973). *The Quantum Theory of Light*, Clarendon Press, Oxford.
Lynch, R. (1995). *Physics Reports* **256**, 367.
Mogilevtsev, D. and Ya, S. K. (1996). *Optics Communications* **132**, 452.
Monroe, C., Meekhof, D. M., King, B. E., and Wineland, D. J. (1996). *Science* **272**, 1131.
Moya-Cessa, H. (1995). *Journal of Modern Optics* **42**, 1741.
Moya-Cessa, H. and Knight, P. L. (1993). *Physical Review A* **48**, 2479.
Obada, A.-S. F. and Abd Al-Kader, G. M. (1998). *Acta Physica Slovaca* **48**, 583.
Obada, A.-S. F. and Abd Al-Kader, G. M. (1999). *International Journal of Modern Physics B* **13**, 2299.

Obada, A.-S. F. and Omar, Z. M. (1995). *ICTP Publications Trieste*. IC/95/302.

Obada, A.-S. F. and Omar, Z. M. (1997). *Physics Letters A* **227**, 349.

Perina, J. (1984). *Quantum Statistics of Linear and Non-Linear Optical Phenomena*, Reidel, Dordrecht, Holland.

Perinova, V., Lukš, A., and Peřina, J. (1998). *Phase in Optics*, World Scientific, Singapore.

Quantum phase and phase dependent measurements (1993). *Physica Scripta* T **48**, 1–142. (Special Issue).

Roy, S. M. and Singh, V. (1982). *Physical Review D: Particles and Fields* **25**, 3413.

Satyanarayana, M. V. (1985). *Physical Review D: Particles and Fields* **32**, 400.

Schleich, W. and Raymer, M. (1997). *Quantum state preparation and measurement*. [Special Issues] *Journal of Modern Optics* **44**, **11**, **12**.

Shor, P. W. and Preskill, J. Quant-ph/0003004, (May 12-2000).

Shore, B. W. and Knight, P. L. (1993). *Journal of Modern Optics* **40**, 1195.

Steinbach, J., Twamley, J., and Knight, P. L. (1997). *Physical Review A* **56**, 4815.

Susskind, L. and Glogower, J. (1964). *Physics* **1**, 49.

Vogel, W. and de Matos Filho, R. L. (1995). *Physical Review A* **52**, 4214.

Walls, D. F. and Milburn, G. J. (1994). *Quantum Optics*, Springer, Berlin.

Wineland, D. J., Monroe, C., Itano, W. M., Leibfried, D., King, B. E., and Meekhof, D. M. (1998). *Journal of Research of National Institute for Standards and Technology* **103**, 259.

Wunsche, A. (1991). *Quantum Optics* **3**, 359.

Wünsche, A. (1998). *Acta Physica Slovaca* **48**, 385.

Yurke, B. and Stoler, D. (1986). *Physics Letters A* **57**, 13.

Zheng, B. and Guo, G.-C. (1996). *Quantum Semiclassical Optics* **8**, 951.